Homework assignment 1 due Monday June 27th.

Probelms from the book.
Section 1.1: 4.
Section 1.2: 7.
Section 1.3: 11, 12, 18, 24.
Section 1.4: 14.
Section 2.3: 1, 13, 20.
Section 1.5: 8, 14, 16, 36.
Section 3.3: 6, 10, 17, 24.
Section 3.4: 1abc, 3abc, 9, 10, 11, 12.

Extra credit problems from the book.
Section 1.3: 34.
Section 1.4: 32.
Section 1.5: 45.
Section 3.3: 39.
Section 3.4: 19.

Problem 1.

The sequence of Fibonacci numbers is defined by recursive relation

$$f_{n+1} = f_n + f_{n-1},$$

and initial conditions

$$f_1 = 1, \quad f_2 = 1.$$

The first few terms of this sequence are $1, 1, 2, 3, 5, 8, 13, 21....$

a. Compute the first 50 terms of the sequence in Sage and plot them.

We want to find the order of growth of that sequence when $n \to \infty$. Notice that your sequence looks like exponentially increasing. Thus, we can make a following conjecture:

Conjecture. There are constants A and α, such that $f_n \sim A\alpha^n$ for large n.

(Reminder: We say that $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.)

b. Prove that if the conjecture is true, then $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \alpha$.

Note that the converse of part **b.** is not true. Therefore, we can't prove the conjecture even if we know the existence of the limit $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$. Still, the existence of that limit is a good *indication* that the conjecture is true.

c. Plot first 50 values of the sequence $\frac{f_{n+1}}{f_n}$ in Sage and estimate the numerical value of α.

Problem 2.

Now, let's try to find α theoretically. In order to do that, we want to find explicit formula for the n-th Fibonacci number f_n. First, let's forget about initial conditions and guess the form of f_n based on just the recursive relation. (If you need help with this problem, see section 1.4 of the book. In particular, theorem 1.7 and exercise 41 should be handy.)

a. Consider the sequence $g_n = \lambda^n$. Find all possible values of λ so that g_n satisfies the recursive relation $g_{n+1} = g_n + g_{n-1}$.

You should get two nonzero values. Call the positive one α and the negative one β.

b. Prove that any linear combination $g_n = A\alpha^n + B\beta^n$ satisfies the recursive relation $g_{n+1} = g_n + g_{n-1}$.

Now we can account for initial conditions:

c. Find the values of A and B so that $g_1 = 1$ and $g_2 = 1$.

d. Show that $f_n = g_n$. That is, prove that if there are two sequences satisfying relations $f_1 = 1,\ f_2 = 1,\ f_{n+1} = f_n + f_{n-1}$, they are necessarily equal.

So we got an explicit formula for f_n. Finally,

e. Prove the existence of constants A and α, such that $f_n \sim A\alpha^n$ for large n.

Problems 3. Perform addition with arbitrary large integers in binary representation. That is, given two strings of zeros and ones of arbitrary length, write a function "Sum" in Sage that returns the sum of those binary numbers. See example 2.5 in the book for reference.

Example: Sum('1010', '11') = '1101'

Example: Sum('1010', '111') = '10001'